The Traveling Firefighter Problem

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ACDA21, July 21st, 2021

Outline

- [8 min] Introducing L_p-TSP
  - A new objective, to be potentially more efficient, fair, unified, …
  - Exact optimization can be hard, even for trees!
  - Let’s approximate ☺
- [6 min] Reduction to Segmented-TSP
  - Enabling PTAS for:
    - Unweighted Euclidean metric
    - Weighted trees
- [6 min] General Metrics
  - Simultaneous 8 approximation for all-norm-TSP
  - 5.65 approximation for Traveling Firefighter Problem (i.e., L_2-TSP)
- Open Problems

"Optimal" Routing

Input: the origin, a set of destinations, and the underlying distances.

Output: an order/permutation to visit the destinations, starting at the origin.

Objective: minimize,

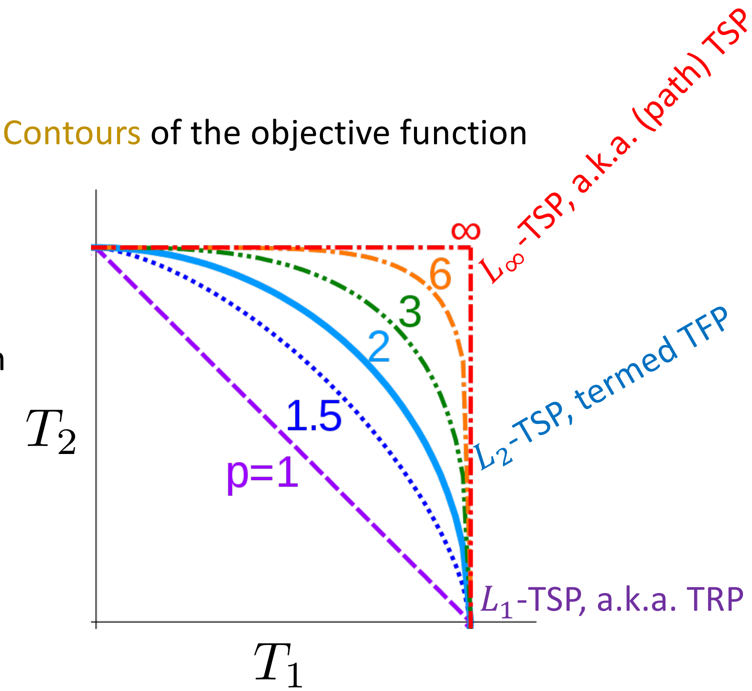
- The latest visit time, equivalently total distance to travel
  - Traveling Salesperson Problem [1832]
- The average/sum of visit times
  - Traveling Repair/Delivery-person Problem [Afrati et al. '85]
- A *norm* of the visit times, e.g.,
- L_2-norm of the visit times, equiv. Sum of Squares of visit times
  - Traveling Firefighter Problem

L_p-TSP

- Objective: minimize the p-norm of the visit times $T: V \to \mathbb{R}$.

$$\|T\|_p := \left(\sum_{v \in V} |T_v|^p\right)^{\frac{1}{p}}$$

Contours of the objective function



Interpolated other combinatorial optimization problems:

- Set Cover vs Min Sum Set Cover [Feige, Lovász, Tetali '04, Golovin-Gupta-Kumar-Tangwongsan '08, Bansal-Batra-Farhadi-Tetali '21]

Applications

Unified Theory + New Algorithms

Ride-sharing

Scheduling

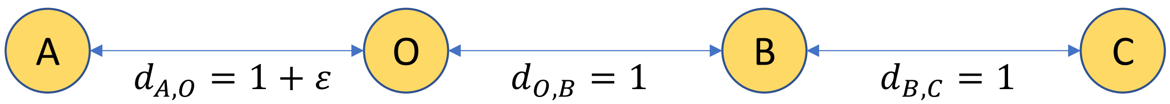
Wildfires & Pandemics

Efficiency

Fairness

Robustness

Does the objective affect the solution?



σ: **Route**	T: **Visit Times**	$\|T\|_\infty$	$\|T\|_1$	$\|T\|_2$
OABC	$(\mathbf{1} + \boldsymbol{\varepsilon}, \mathbf{0}, \mathbf{3} + \mathbf{2}\boldsymbol{\varepsilon}, \mathbf{4} + \mathbf{2}\boldsymbol{\varepsilon})$	$\mathbf{4} + \mathbf{2}\boldsymbol{\varepsilon}$	**8+5**$\boldsymbol{\varepsilon}$	$\sqrt{26 + 30\varepsilon + 9\varepsilon^2}$
OBCA	$(\mathbf{5} + \boldsymbol{\varepsilon}, \mathbf{0}, \mathbf{1}, \mathbf{2})$	$\mathbf{5} + \boldsymbol{\varepsilon}$	**8+**$\boldsymbol{\varepsilon}$	$\sqrt{30 + 10\varepsilon + \varepsilon^2}$

$$\|T\|_p := \left(\sum_{v \in V} |T_v|^p \right)^{\frac{1}{p}}$$

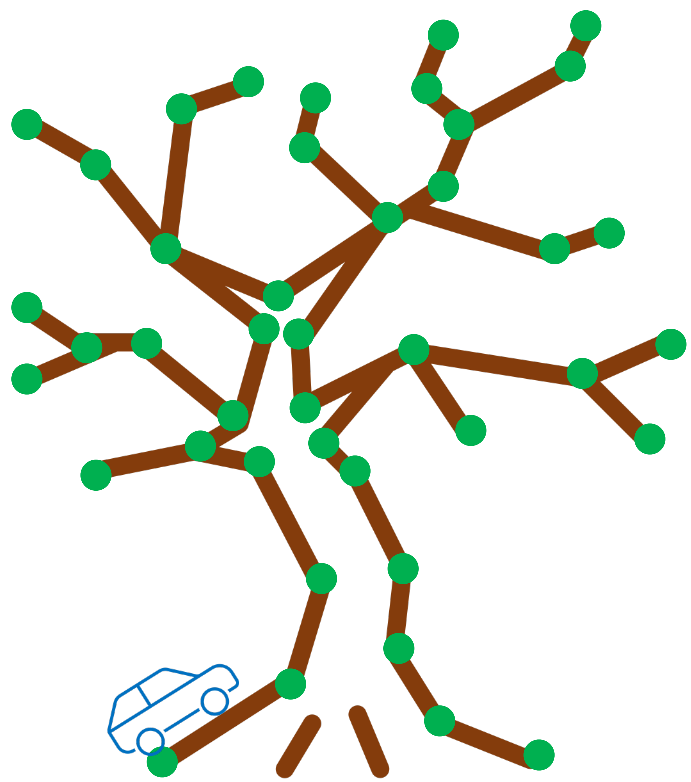
Does the objective affect complexity?

- Yes
  - even for trees

  - $p = \infty$: Linear Time Solvable

  - $p = 1$: Strongly NP-hard [Sitters '02]
    - ✓ PTAS [Sitters '14]

  - ○ $p \in (1, \infty)$



Active Extensive Literature

	TSP (L_∞-TSP)	TRP (L_1-TSP)
Constant factor approximated on general metrics	**3/2** [Christofides '76, Serdyukov '78] **3/2-1e-36** [Karlin-Klein-Oveis Gharan '21]	**O(1)** [Blum-Chalasani-Coppersmith-Pulleyblank-Raghavan-Sudan '94] **7.18** [Goemans-Kleinberg '96] **3.59** [Chaudhuri-Godfrey-Rao-Talwar '03]
Inapproximable for general metrics within	**1 + 1/122** [Karpinski-Lampis-Schmied '15]	
$1+\varepsilon$ approximation	**Euclidean** [Arora '96, Mitchel '96] **Planar** [Grigni-Koutsoupias-Papadimitriou '95, Arora-Grigni-Karger-Klein-Woloszyn '98, Klein '05] **Path ⇒ Tour** [Traub-Vygen-Zenklusen '19]	**Line metric** [Afrati-Cosmadakis-Papadimitriou-Papageorgiou-Papakostantinou '86] **Weighted trees,Euclidean plane** [Sitters '14]

L_p-TSP ⇒ Segmented-TSP

Theorem 1 [FTT]: There is a PTAS for L_p-TSP on weighted tree-metrics, and unweighted 2D-Euclidean.

Reduction to k-TSP enables a QPTAS
- Generalizing the result of [Archer-Williamson '03] for TRP
- Requiring $O(\varepsilon^{-1}\log n)$ TSP sub-routes
- For a PTAS, we need a more general sub-problem

Idea: DP reduction to Segmented-TSP [Sitters '14], at the cost of $(1+\varepsilon)$ multiplicative error.

Segmented-TSP:
- A generalization of k-TSP
- Given k deadlines, $t_1, \ldots, t_k$, and numbers $n_1 \leq \cdots \leq n_k$
- Decide whether it is possible to visit n_i vertices by time $t_i, \forall i$
- If the answer is yes, an α-approx. solution is a solution for $\{\alpha t_i\}_{[k]}, \{n_i\}_{[k]}$
- Segmented-TSP has PTAS for tree as well as Euclidean metrics [Sitters '14]

The Reduction

- Discovering structure in (approximately) optimal solutions
  - Quantizing distances, allows the following, WLOG

$$d(i,j) \in \{0\} \cup [O(n^2/\varepsilon)] = \{0\} \cup [\tilde{O}(n^2)] \quad \forall i,j$$

    - Enables breaking into $\gamma = O(\log n \cdot \varepsilon^{-1})$ shortest paths between time spots

$$1, (1+\varepsilon), \cdots, (1+\varepsilon)^{\gamma}$$

  - Returning to the origin to further reduce the number of segments to

$$k = O(1+\varepsilon^{-2}) \qquad c := (1+\varepsilon)^k \geq 3$$

    - At times $\lambda_i := (1+\varepsilon)^{-j} \cdot c^i, \quad \forall i \geq 0$

The Reduction (ctnd.)

- Generalizing the result by [Sitters '14]. For any L_p-TSP,

Lemma [FTT]: ∃ a near optimal route, visiting new vertices during $[3\lambda_{i-1}, \lambda_i]$, remaining at the origin until $3\lambda_i$.

- Finding the best structured solution through Dynamic Programming
  - $DP[i, d]$, assuming d vertices are visited up to λ_i, stores their minimum possible contribution to the objective
  - Update: considering $O(n^k)$ cases
    - corresponding to the # of vertices visited up to

$$\lambda_{i-1}, 3\lambda_{i-1} + \lambda_i \cdot (1+\varepsilon)^{-k}, \cdots, 3\lambda_{i-1} + \lambda_i$$

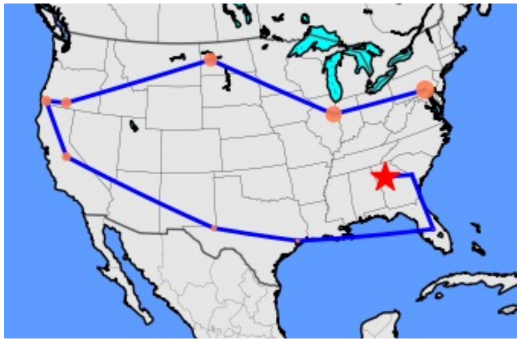
    - Each defines be a segmented-TSP instance with $k = O(1 + \varepsilon^{-2})$

Lemma [FTT]: Any α approximation for Segmented-TSP enables a $(1+\varepsilon)\alpha$ approximation for L_p-TSP.

Cost of the optimal solution to a wrong problem

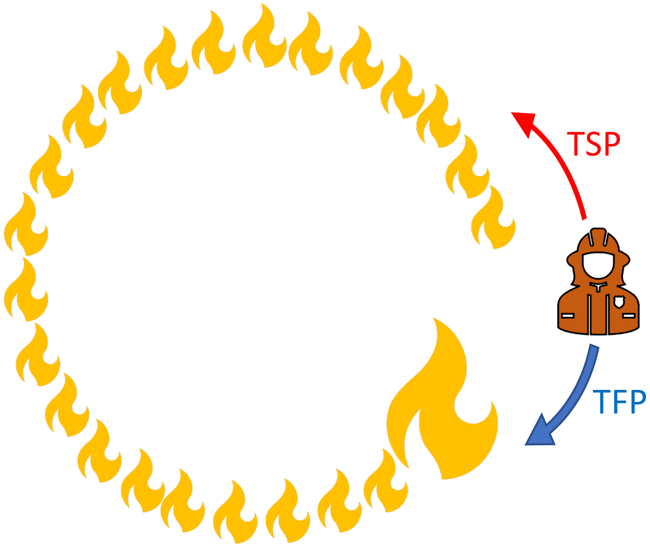
- L_∞-TSP

Major wildfires in Dec 2019



- L_2-TSP

Can be (multiplicatively) **unbounded**



All-norm-TSP

- Can we find a route that is approximately optimal with respect to any norm of the visit times vector?

$$\min_{\sigma} \sup_{\|\cdot\|} \frac{\|T_\sigma\|}{\min_{\sigma' \in} \|T_{\sigma'}\|}$$

  - [Golovin-Gupta-Kumar-Tangwongsan '08] introduced a 16-approximation.

Theorem 2 [FTT]: We can find a route that is, simultaneously, 8-approximate with respect to any norm of T.

Idea: Partial Covering + a mild relaxation of k-MST

The Partial Covering Algorithm

- Introduced by [Blum-Chalasani-Coppersmith-Pulleyblank-Raghavan-Sudan '94] for TRP
- Developed through [Goemans-Kleinberg '98, Chaudhuri-Godfrey-Rao-Talwar '03]
- Leads to 16-approximate all-norm-TSP [Golovin-Gupta-Kumar-Tangwongsan '08]
  - $b = 1$ (WLOG, the distance of the closest destination to the origin)
  - $c = 2$

1: **procedure** GEOMETRIC-COVERING(V, s, d)
2: Algorithm Parameters: $b \in (0, \infty), c \in (1, \infty)$
3: $i \leftarrow 0$
4: **while** there remains destinations to visit **do**
5: $\triangleright$ Conducting sub-tours
6: $C_i \leftarrow$ a maximal route of length $\leq b \cdot c^i$.
7: Travel through C_i (and return to the origin)
8: $i \leftarrow i + 1$
9: **return** an ordering σ of V according to their (first) visit time through the above loop.

In line-6, we use (DFS of) a *good-k-tree*:
- [Chaudhuri-Godfrey-Rao-Talwar '03]
- a mild relaxation of k-MST
- i.e., not larger than a k-TSP / k-stroll
- found using a primal-dual method

Steps in the Analysis

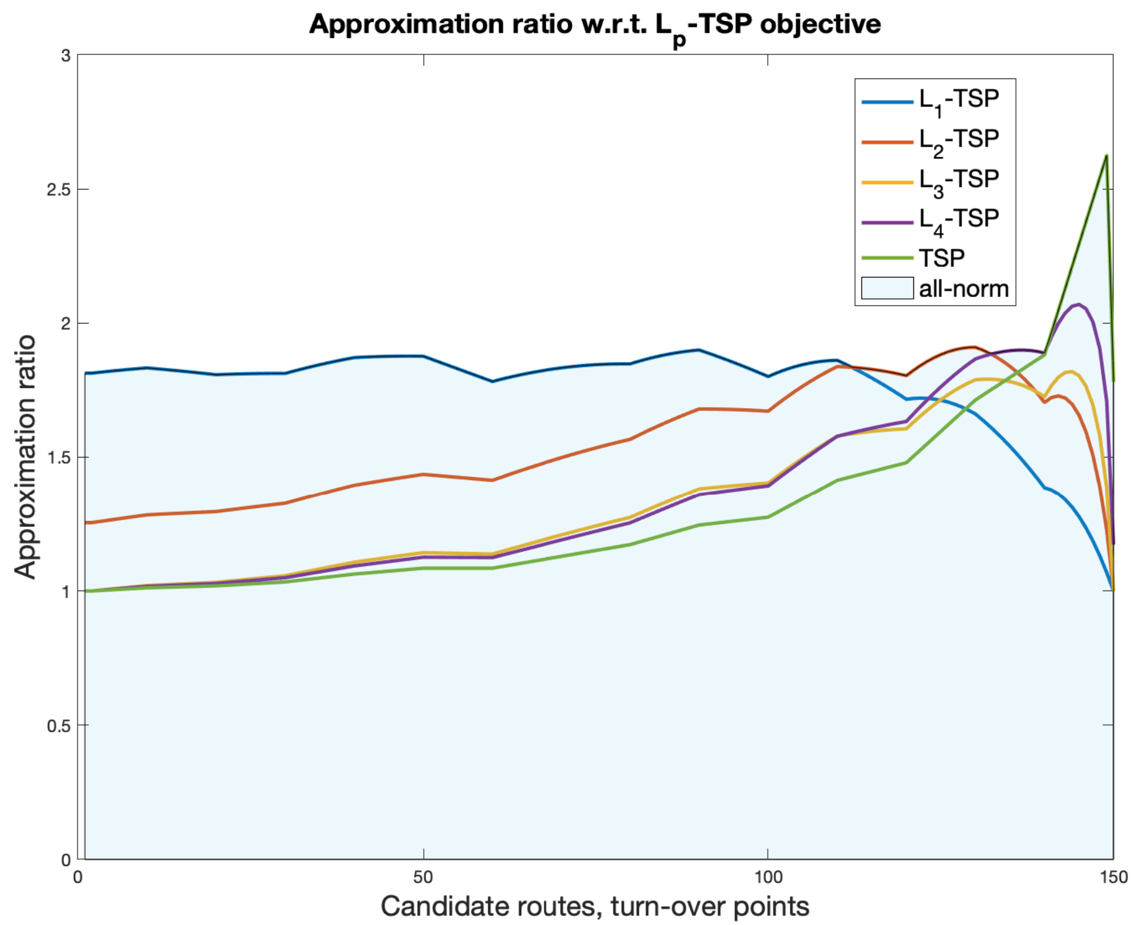
- Let $T_k^{\mathrm{OPT}} \in [2^i, 2^{i+1})$
- There is a k-path, no longer than 2^{i+1}
- We have a good-k-tree, of total length no more than 2^{i+1}
- Hence, C_{i+1} has at least k vertices
- Finally, we have 8-submajorization of the optimal route by ALG.

$$T_k^{\mathrm{ALG}} \leq \sum_{j=0}^{i+1} |C_j| \leq \sum_{j=0}^{i+1} 2 \times 2^j < 2^{i+3} \leq 8T_k^{\mathrm{OPT}}$$

Approximate All-norm-TSP can be impossible

Theorem 3 [FTT]: Guaranteeing an α-approximate all-norm-TSP, for line metric, is impossible for $\alpha \leq 1.78$.

- Starting at $x = 0$, and destinations at $\{-1\} \cup \{b^i - 1 : i \in [n]\}$
- This gives $\alpha \geq 1.67$ for $b = 1.001, n = 2100$
- The following numerical example, w/ similar structure, gives $\alpha \geq 1.78$


Approximation ratio w.r.t. L_p-TSP objective
L_1-TSP
L_2-TSP
L_3-TSP
L_4-TSP
TSP
all-norm
Approximation ratio
Candidate routes, turn-over points
0
0.5
1
1.5
2
2.5
3
0
50
100
150

Improved approximation for TFP

- Choosing $b = c^U$, with U uniformly distributed over $[0,1]$

$$\frac{\mathbb{E}[(T_k^{\mathrm{ALG}})^2]}{(T_k^{\mathrm{OPT}})^2} \leq \frac{c+1}{c-1} \cdot 2c^2 / \ln c$$

- Optimizing, for $c \simeq 2.54$ we have

$$\mathbb{E}[\|T^{\mathrm{ALG}}\|_2^2] \leq 31.82 \|T^{\mathrm{ALG}}\|_2^2$$

- Resulting $\sqrt{31.82} \simeq 5.641$ approximation for TFP.

Theorem 4 [FTT]: TFP can be 5.65 approximated on general metrics.

Problems

- Unified algorithms for L_p TSP Problems?
- [in]approximability, e.g., what is the hardest L_p TSP?
- Multiple Vehicle/Depots
- Online problem
- Verifying the best norm for containing wildfires, pandemics, etc.

Thank you for joining!